Journal of Statistical Physics, Vol. 72, Nos. 1/2, 1993

Some Rigorous Results on Majority Rule Renormalization Group Transformations near the Critical Point

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Received September 29, 1992

We consider the majority rule renormalization group transformation with two-by-two blocks for the Ising model on a two-dimensional square lattice. For three particular choices of the block spin configuration we prove that the model conditioned on the block spin configuration remains in the high-temperature phase even when the temperature is slightly below the critical temperature of the ordinary Ising model with no conditioning. We take as the definition of the infinite-volume limit an equation introduced in earlier work by the author. We use a computer to find an approximate solution of this equation and verify a condition which implies the existence of an exact solution.

KEY WORDS: Lattice spin system; majority rule; rigorous high-temperature phase.

1. INTRODUCTION

Position space renormalization group transformations are a powerful technique for studying the critical behavior of lattice spin systems with Ising-like spins.[(11)] A widely used example of such a transformation is the majority rule transformation. In this transformation the block spin is taken to be $+1$ or -1 according to whether the majority of spins in the block are up or down. Although these methods have been used extensively in the numerical study of second-order phase transitions, there are essentially no rigorous results about these renormalization group maps in a neighborhood of the critical point. In this paper we will make a small step toward improving this situation.

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Given a measure on the configurations of the original spins, a blocking procedure like majority rule yields a measure on the configurations of block spins. If the measure on the original spins σ is a Gibbs measure for a Hamiltonian $H(\sigma)$, then one would like to show that the measure on the block spins s is the Gibbs measure of some new Hamiltonian $\tilde{H}(s)$. Formally this new Hamiltonian is given by

$$\exp[\tilde{H}(s)] = \sum_{\sigma} T(s, \sigma) \exp[H(\sigma)] \tag{1.1}$$

In majority rule transformations, $T(s, \sigma)$ is 1 if the majority of the spins in each block agree with the block spin in that block. Otherwise it is 0. More generally, one can consider renormalization group transformations for other probability kernels $T(s, \sigma)$ which satisfy

$$\sum_{s} T(s, \sigma) = 1 \tag{1.2}$$

This condition implies that $\sum_{\sigma} e^{H(\sigma)} = \sum_{s} e^{\tilde{H}(s)}$.

It is important to emphasize that Eq. (1.1) is only formal. In a finite volume it makes sense, but then one must take an infinite-volume limit. The existence of this infinite-volume limit has been rigorously established only in the case of very high temperature or very large magnetic field.[5,7,8] Griffiths and Pearce[4-6] and Israel[7] argued that at low temperatures the definition of this renormalization group map has serious problems. Van Enter *et al.*[13] used their ideas to prove that the renormalization group map is not defined at low temperatures for a variety of specific maps. More precisely, they show that the measure on the block spins is not the Gibbs measure of any Hamiltonian in a large Banach space of Hamiltonians. Their examples include the majority rule transformation with 7-by-7 blocks (and other larger block sizes). One can object that the measure for the block spins may be Gibbsian for a Hamiltonian in a larger Banach space of Hamiltonians than the space they considered, but this would not be a particularly satisfying resolution. Part of the renormalization group picture is that while the renormalization group map produces a Hamiltonian with an infinite number of different types of terms and so forces one to work in an infinite-dimensional space of parameters, these parameters should fall off rapidly as the number of sites in the term or the distances involved in the term increase. If the Hamiltonian for the block spins is outside the space of Hamiltonians considered in ref. 13, then this part of the renormalization group picture would be in trouble.

The rigorous results of ref. 13 are for low temperatures (low-temperature expansions are involved, so low here means well below the critical temperature). Thus there are serious problems with using the majority rule renormalization group to study first-order phase transitions. These results do not say that the majority rule renormalization group map is not defined in a neighborhood of the second-order transition. We should also point out that at very low temperatures renormalization group maps that are quite different from Eq. (1.1) have been rigorously defined and studied.[3]

A key tenet of the renormalization group philosophy is that the renormalization group map is smooth in a neighborhood of the critical point, even though quantities like the free energy have singularities. (Ref. 13 proved that if the renormalization group map is defined, then it is at least Lipschitz continuous. At the moment this result says nothing about the vicinity of the critical point, since the map has not been proven to be defined there.) To see why this smoothness is at least plausible, consider Eq. (1.1) and take the Hamiltonian $H(\sigma)$ to be the usual Ising Hamiltonian at the critical temperature. If the sum over σ in Eq. (1.1) were over all σ, then we would have a singularity on our hands. But the sum is not over all σ, it is conditioned on the block spin configuration. The hope is that this conditioning on the block spins moves the model off the critical point, so that the conditioned partition function in the right side of Eq. (1.1) is not critical. In other words, fix a particular block spin configuration s and consider the probability measure on the original spins σ associated with the partition function in Eq. (1.1). The expectation of a functional $F(\sigma)$ of the spins σ is given by

$$\langle F(\sigma)\rangle_s = Z^{-1} \sum_{\sigma} F(\sigma)\, T(s, \sigma) \exp[H(\sigma)] \tag{1.3}$$

(Z is determined by the requirement that this is a probability measure, and will depend on s.) One would like to show that even if the Hamiltonian $H(\sigma)$ is the Ising Hamiltonian at the critical point, for a fixed block spin configuration s, the measure $\langle \cdot \rangle_s$ is not critical. In particular, we expect it to have a finite correlation length even when the unconditioned system has an infinite correlation length.

We prove that this is indeed the case for the two-dimensional Ising model for the three particular choices of the block spin configuration shown in Fig. 1. For the first block spin configuration (Fig. 1a), in which all the spins are $+$, this is no great surprise. Conditioning on this block spin configuration is like applying a magnetic field to the system, since the block spins favor a majority of the spins in each block being $+$ rather than $-$. For the other two block spin configurations (Fig. 1b and 1c), our result is less trivial. Indeed, for the checkerboard block configuration (Fig. 1b), it is expected that $\langle \cdot \rangle_s$ will undergo a phase transition at low enough temperature. What our result shows is that the associated critical

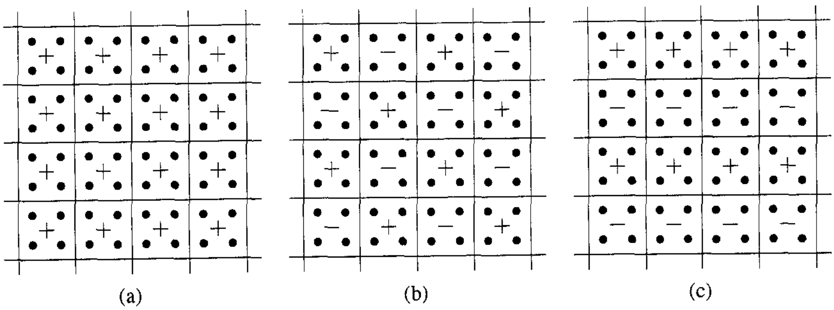

Fig. 1. The three block spin configurations considered: (a) Ferromagnetic configuration, (b) checkerboard configuration, (c) strip configuration. The block spins are marked with a + or a −, while the original spins are indicated by a •.

temperature is strictly lower than the critical temperature of the ordinary Ising model with no conditioning on a block spin configuration.

To prove this result, we need a method for treating systems which are in the high-temperature phase, but whose temperature is not sufficiently high that techniques like high-temperature expansions or the Dobrushin uniqueness theorem can be applied. Such a method was developed by Dobrushin and Shlosman.[2] They found a sequence of finite-volume conditions, any one of which implies that the model is in the high-temperature phase. For a given volume the conditions may be checked on a computer. (Some examples of this are refs. 1 and 12.) Moreover, it is expected that for most discrete spin systems, if the temperature is above the critical temperature, then their finite-volume condition should be satisfied for sufficiently large volumes. Of course the amount of calculation required to show that a particular system satisfies one of the conditions may be prohibitive. We have not attempted to apply their method to the problem at hand.

We use the method developed in ref. 9. In this method we forgo the usual definition of the infinite-volume limit and instead define the infinite-volume limit as the solution of a certain fixed-point equation. If the temperature is sufficiently high, one can prove that this fixed point equation definition of the infinite-volume limit agrees with the usual definition. (We give a sketch of the proof using high-temperature expansions in the next section. One can also prove this using the Dobrushin–Shlosman techniques.[10]) One very nice feature of this method is that the number of dimensions is effectively reduced by one. In the two-dimensional models we consider here this means that the fixed-point equation only involves the sites in what is effectively a one-dimensional set.

While we can prove that our definition of the infinite-volume limit agrees with the usual one at very high temperatures, we cannot prove this for all temperatures. Hence we will distinguish the two definitions by referring to the free energy and correlation functions that come out of our fixed-point equation as the fixed-point-equation free energy and correlation functions. For the region of β in which we prove that our fixed-point equation has a solution, we also show that the fixed-point equation free energy and correlation functions are analytic in β. This of course implies that the two definitions agree in any connected subset of this region in which the usual free energy and correlation functions are analytic. In fact it implies a little more. Suppose that with the usual definition of the infinite-volume limit we have a phase transition at $\beta = \beta_c$, meaning that the usual free energy is analytic in $[0, \beta_c)$ but not in a neighborhood of β_c. Suppose also that we have shown that the fixed-point-equation quantities are analytic in an open region containing the interval $[0. \beta_c]$. Then the phase transition cannot be a typical second-order transition in which derivatives of the free energy diverge as $\beta \to \beta_c$ from the high-temperature side, since these derivatives must agree with the derivatives of the fixed-point equation free energy on the high-temperature side of β_c, and the latter do not diverge. Thus if we can show that with the fixed-point equation definition of the infinite-volume-limit the various quantities are analytic from 0 to β_0, then with the usual definition of the infinite-volume limit if we assume that the first phase transition is a typical second-order transition, then it must occur at a β at least as large as β_0.

Our theorem concerns the majority rule with two-by-two blocks. In this case the block may contain equal numbers of plus and minus spins, so we must explain what we do in the case of such ties. We take the probability kernel to be

$$T(s, \sigma) = \prod_B t(s_B, \{\sigma_i\}_{i \in B})$$

where the product is over the two-by-two blocks B, s_B is the block spin for that block, and $\{\sigma_i\}_{i \in B}$ are the four original spins in the block. The kernel $t(\cdot)$ for a single block is equal to 1 if three or more of the four original spins in the block agree with the block spin, equal to 0 if three or more disagree with the block spin, and equal to 1/2 if two of the original spins are plus and two are minus and the block spin is either plus or minus. This kernel is explicitly given by

$$\begin{aligned} t(s_B; \sigma_0, \sigma_1, \sigma_2, \sigma_3) \\ &= \tfrac{1}{2} + \tfrac{3}{16} s_B(\sigma_0 + \sigma_1 + \sigma_2 + \sigma_3) \\ &- \tfrac{1}{16} s_B(\sigma_1 \sigma_2 \sigma_3 + \sigma_0 \sigma_2 \sigma_3 + \sigma_0 \sigma_1 \sigma_3 + \sigma_0 \sigma_1 \sigma_2) \end{aligned}$$

where $\sigma_0, \sigma_1, \sigma_2, \sigma_3$ are the original spins in the block B.

Theorem. Let $H(\sigma)$ be the Hamiltonian of the usual Ising model on the two-dimensional square lattice. Let β_c be the critical inverse temperature of this model $[\beta_c = \frac{1}{2}\log(1+\sqrt{2}) = 0.4407...]$. There is a $\beta_0 > \beta_c$ such that if s is one of the three block spin configurations shown in Fig. 1, then the fixed-point-equation free energy and correlation functions for the model defined by Eq. (1.3), i.e., conditioned on this block spin configuration, are analytic in a neighborhood of $[0, \beta_0]$ and agree with the usual free energy and correlation functions in a neighborhood of $\beta = 0$. Moreover, the fixed-point equation truncated correlation functions have exponential decay with a correlation length that is bounded as β ranges from 0 to β_0.

Remarks. 1. Our proof of the theorem is computer assisted in the following sense. First we prove that if a certain condition that only involves a finite set of sites is satisfied, then the conclusions of the theorem hold. Then we use the computer to check if this condition is satisfied. However, we do not use interval arithmetic in these calculations. This is a bit of a cheat, but we should stress that the numerical calculations are not ill behaved and while we do not have rigorous bounds on the numerical errors, we have checked that these errors are insignificant.

2. For some choices of the block spin configuration s the conditioned system may have a phase transition with a diverging correlation length at some $\beta_c(s)$. What the theorem says is that for the three block spin configurations is Fig. 1, $\beta_c(s) > \beta_c$.

3. The computer calculations are carried out at β_c, the critical temperature of the unconditioned model. The finite-volume condition is such that if it holds at a particular temperature, then it holds in a neighborhood of that temperature. Hence, verifying that the condition holds at β_c implies that it holds for β slightly greater than β_c, although we do not get an explicit bound on β_0. We could obtain an explicit bound on β_0 by carrying out the calculations at β's slightly larger than β_c. The resulting bound would be quite close to β_c. The finite-volume condition is just barely satisfied at β_c. Of course, if one can actually *prove* that a system is in the high-temperature phase at a particular temperature, then the actual critical temperature is probably a good bit lower. In other words, we expect the critical temperatures for the conditioned systems to be significantly lower than the critical temperature of the unconditioned system.

It would be interesting to estimate numerically the critical temperature of the conditioned measure for some simple block spin configurations like those shown in Fig. 1. For example, one could use Padé approximants to study the high-temperature series of these models. Since we can actually prove that the critical temperature is lower than the critical temperature of the unconditioned model, we expect that it is lower by a fair amount.

Obviously, what we have done in this paper is a very small step toward the goal of putting transformations like the majority rule on a rigorous footing in a neighborhood of the critical point. The next step from our point of view is to prove that the theorem we have proved here for three particular block spin configurations is true for all block spin configurations with a $\beta_0 > \beta_c$ which may be taken to be independent of the block spin configuration. It should be noted that in this approach we are trying to prove more than is absolutely necessary for the majority rule map to be defined. If there is a set of block spin configurations for which the theorem is not true, but this set has measure zero, then the definition of the majority rule map could still be acceptable. Showing that the theorem is true for all block spin configurations does not complete the first step of the renormalization group transformation. We must still show the existence of the Hamiltonian $\tilde{H}(\sigma)$, and hopefully some nice decay properties for this Hamiltonian.

All of the above is only concerned with the first step of the renormalization group. The ultimate goal is to show that there is an open set of Hamiltonians on which the map is rigorously defined which contains a fixed point of the map. Passing through this fixed point should be a stable manifold that contains the usual Ising Hamiltonian at the critical point. (At this point in the discussion there are two fixed-point equations. The first is merely used to define the infinite-volume limit. Finding a solution for this fixed-point equation amounts to a single iteration of the renormalization group map. The second fixed-point equation is for the renormalization group map itself. This is the fixed point that the Hamiltonian of a critical model should converge to in the limit of an infinite number of iterations of the renormalization group map. Throughout this paper we are only concerned with the first iteration of the renormalization group map, and so the fixed-point equation we discuss will always refer to the first fixed-point equation.)

In Section 2 we derive the fixed-point equation that we use to define the infinite-volume limit. This derivation is heuristic, but we sketch how one may use high-temperature expansions to prove that for sufficiently high temperatures this definition of the infinite-volume limit agrees with the usual one. The derivation closely parallels the derivation in ref. 9 of a similar fixed-point equation for the ordinary Ising model with no majority rule transformation. In ref. 9 we proved that a certain condition involving an approximate solution of the equation implies the existence of an exact solution. An analogous result holds for our fixed-point equation. We state the result and refer the reader to ref. 9 for its proof. The techniques of ref. 9 also show that this condition implies analyticity of various quantities, e.g., the free energy and correlation functions, in parameters such as the temperature. In Section 3 we turn to the question of finding approximate solutions to the fixed-point equation that satisfy the condition. Checking that the condition holds requires a fair amount of computer calculation, so it is important to develop algorithms that are reasonably efficient. We briefly discuss some of the tricks that we have found to be useful. We then present the results of the numerical calculations that show that for the three block spin configurations of Fig. 1, we can find approximated fixed points that satisfy the condition.

2. A FIXED-POINT EQUATION FOR THE INFINITE-VOLUME LIMIT

Throughout this section we make use of the following fact. If $f(\sigma)$ is a function of the spins σ_i, where i ranges over some finite set V, then $f(\sigma)$ may be written in a unique way as

$$f(\sigma)=\sum_X c(X)\,\sigma(X)$$

where $\sigma(X)=\prod_{i\in X}\sigma_i$ and X is summed over all subsets of V. The numbers $c(X)$ are given by

$$c(X)=2^{-|V|}\sum_\sigma \sigma(X)\,f(\sigma)$$

where $|V|$ is the number of sites in V, and the sum is over all spin configurations on V.

Consider the volume Λ shown in Fig. 2. The partition function for this volume is

$$Z_\Lambda=\sum_{\{\sigma_i\}_{i\in\Lambda}}\prod_{B\subset\Lambda} t(s_B,\{\sigma_i\}_{i\in B})e^{H_\Lambda}$$

We include in the Hamiltonian H_Λ the terms that couple spins inside Λ to spins outside of Λ. Thus the partition function Z_Λ will be a function of the spins in the boundary $\partial\Lambda$, as well as the block spins s_B with $B\subset\Lambda$. (A site is in $\partial\Lambda$ if it is not in Λ, but it has a nearest neighbor that is.) We will first consider the block spin configuration in which all $s_B=+1$, and suppress the dependence of Z_Λ on s_B. Since Z_Λ is always positive, we can write it in the form

$$Z_\Lambda=\exp\left[\sum_X c_\Lambda(X)\,\sigma(X)\right] \tag{2.1}$$

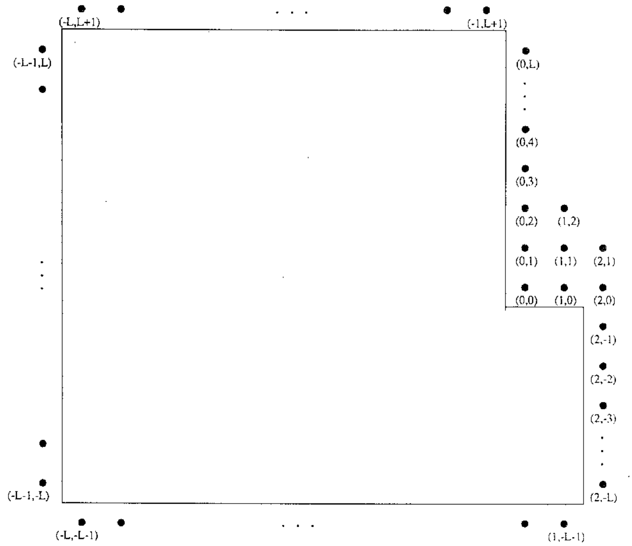

Fig. 2. The volume Λ and some of the sites in $\partial\Lambda$. The volume Λ' is obtained by including the four sites (0, 0), (0, 1), (1, 0), and (1, 1).

where X is summed over subsets of $\partial\Lambda$. Now consider the volume Λ' which is obtained from Λ by adding the two-by-two block $B_0 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

Again, we can write

$$Z_{\Lambda'} = \exp\left[\sum_X c_{\Lambda'}(X)\,\sigma(X)\right] \tag{2.2}$$

where now X is summed over subsets of $\partial\Lambda'$. We can obtain $Z_{\Lambda'}$ from Z_Λ by summing over the sites in the added block,

$$Z_{\Lambda'} = \sum_{\{\sigma_i\}_{i\in B_0}} t(s_{B_0}, \{\sigma_i\}_{i\in B_0}) \exp(H_{B_0}) Z_\Lambda \tag{2.3}$$

The sum over $\{\sigma_i\}_{i\in B_0}$ is over the spin configurations on the block B_0. H_{B_0} contains those terms in the Hamiltonian for Λ' that did not appear in the Hamiltonian for Λ, i.e., terms which couple a site in B_0 to a site in $\partial\Lambda'$. Using (2.1) and (2.2), we find that (2.3) becomes

$$\begin{aligned}
&\exp\left[\sum_{X\subset\partial\Lambda'} c_{\Lambda'}(X)\,\sigma(X)\right] \\
&\quad = \sum_{\{\sigma_i\}_{i\in B_0}} t(s_{B_0}, \{\sigma_i\}_{i\in B_0}) \exp\left[H_{B_0} + \sum_{X\subset\partial\Lambda} c_\Lambda(X)\,\sigma(X)\right]
\end{aligned}$$

If $X \cap B_0 = \varnothing$, then the term $\exp[c_\Lambda(X)\,\sigma(X)]$ can be factored past the sum over $\{\sigma_i\}_{i\in B_0}$. Thus

$$\exp\left[\sum_{X\subset\partial\Lambda'} c_{\Lambda'}(X)\,\sigma(X)\right]$$
$$=\exp\left[\sum_{X\subset\partial\Lambda,\,X\cap B_0=\varnothing} c_\Lambda(X)\,\sigma(X)\right]\exp\left[\sum_{X\subset\partial\Lambda'} f_\Lambda(X)\,\sigma(X)\right] \tag{2.4}$$

where $f_\Lambda(X)$ is defined by

$$\exp\left[\sum_{X\subset\partial\Lambda'} f_\Lambda(X)\,\sigma(X)\right]$$
$$=\sum_{\{\sigma_i\}_{i\in B_0}} t(s_{B_0}, \{\sigma_i\}_{i\in B_0})\exp\left[H_{B_0}+\sum_{X\subset\partial\Lambda,\,X\cap B_0\neq\varnothing} c_\Lambda(X)\,\sigma(X)\right] \tag{2.5}$$

Then (2.4) implies for sets X such that $X\subset\partial\Lambda'$ and $X\subset\partial\Lambda$.

$$cA'(X)=c_\Lambda(X)+f_\Lambda(X) \tag{2.6}$$

For sets X such that $X\subset\partial\Lambda'$ but $X\not\subset\partial\Lambda$, it implies

$$c_{\Lambda'}(X)=f_\Lambda(X) \tag{2.7}$$

All the preceding equations are rigorous since they only involve a finite number of sites. We will consider the following infinite-volume limit of Λ. In Fig. 2 we let $L\to\infty$, so the right side of $\partial\Lambda$ including the kink is kept fixed, while the top, bottom, and left sides move off to ∞. The fixed-point equation that we will derive will only involve $c_\Lambda(X)$ for X contained in the $L\to\infty$ limit of the right side of $\partial\Lambda$. We denote this set of sites by $\partial\Lambda_\infty$. Explicitly,

$$\partial\Lambda_\infty=\{(0,n): n=0,1,2,...\}\cup\{(1,0)\}\cup\{(2,-n): n=1,2,3,...\}$$

We denote by $\partial\Lambda'_\infty$ the analogous set of sites obtained using $\partial\Lambda'$ in place of $\partial\Lambda$. The set $\partial\Lambda'_\infty$ is the same as $\partial\Lambda_\infty$ except that it does not include the sites $(0,1)$, $(0,0)$, and $(1,0)$, while it does include the sites $(1,2)$, $(2,1)$, and $(2,0)$.

If β is sufficiently small, then there is a rigorous high-temperature expansion for $\ln Z_\Lambda$. It yields convergent expansions for the coefficients $c_\Lambda(X)$. From these expansions one obtains the following bound on $c_\Lambda(X)$. Let G denote a set of bonds. We define ∂G to be the set of sites which belong to an odd number of bonds in G. If we did not include the renormalization group kernel $T(s,\sigma)$, i.e., we just considered the ordinary Ising model, then the expansion for $c_\Lambda(X)$ would be over graphs G such that $\partial G = X$. With the kernel $T(s, \sigma)$ included, ∂G may include sites inside Λ. So the graphs that contribute to $c_\Lambda(X)$ will have $X \subset \partial G$. We define

$$n(X) = \min\{|G|: X \subset \partial G, G \text{ is connected}\}$$

where $|G|$ is the number of bonds in G. Then, for $X \subset \partial\Lambda_\infty$, we have

$$|c_\Lambda(X)| \leqslant e^{-mn(X)} \tag{2.8}$$

where m is a positive constant we can make as large as we want by taking β sufficiently small. (m does not depend on Λ.)

Let L_1 and L_2 be two different values of L and consider the difference $c_{\Lambda(L_1)} - c_{\Lambda(L_2)}$. (We have made the dependence of Λ on L explicit here.) The graphs which contribute to this difference come from connected graphs G with $X \subset \partial G$ and such that G extends from X all the way to the top, bottom, or left boundaries of $\Lambda(L_1)$ or $\Lambda(L_2)$. Thus the high-temperature expansion implies

$$|c_{\Lambda(L_1)} - c_{\Lambda(L_2)}| \leqslant e^{-m(L-d(X))}$$

where $L = \min\{L_1, L_2\}$, $d(X)$ is the distance from the origin to the site in X closest to the origin, and m can be made as large as desired by requiring β to be sufficiently small. This estimate implies that $\lim_{L\to\infty} c_{\Lambda(L)}(X)$ exists. We denote this limit by $c(X)$. Note that (2.8) implies the same bound on $c(X)$.

Similarly, $\lim_{L\to\infty} c_{\Lambda'(L)}(X)$ exists. However, this limit is not equal to $c(X)$, since the location of the kink in Λ' is shifted with respect to its location in Λ. Taking this shift into account, we see that

$$\lim_{L\to\infty} c_{\Lambda'(L)}(X) = c(X-(0,2)) \tag{2.9}$$

where $X-(0,2)$ is the set of sites obtained by subtracting $(0,2)$ from each site in X.

The fixed-point equation we will define will only involve $c(X)$ with $X \subset \partial\Lambda_\infty$ and $X \cap B_0 \neq \varnothing$. We will denote $\{c(X): X \subset \partial\Lambda_\infty, X \cap B_0 \neq \varnothing\partial$ by just c. We will introduce a norm on the set of such c. For each site i, let μ_i be a nonnegative number such that $\mu_i = \mu_j$ if i and j occupy the same position within their respective blocks, i.e., $i = j + (2n, 2m)$ for some integers n, m. (Of course this means there are only four different μ_i.) Define

$$\|c\| = \sum_X |c(X)|\, e^{\mu(X)} \tag{2.10}$$

where $\mu(X)=\sum_{i\in X}\mu_i$. Keeping in mind the restriction that the $\mu_i\geqslant 0$, it is easy to show[9] that we have a Banach algebra, i.e., $\|c_1c_2\|\leqslant\|c_1\|\cdot\|c_2\|$. When β is small, the norm with all the $\mu_i=0$ will suffice. To prove our theorem we will need to use nonzero values of μ_i.

We must also consider the infinite-volume limit of $f_\Lambda(X)$. The bounds we have imply that we can make

$$\left\| \sum_{X\subset\partial\Lambda,\, X\cap B_0\neq\varnothing} c_\Lambda(X)\,\sigma(X)\right\|$$

as small as we like by taking β sufficiently small. Thus, for small β we can write $f_\Lambda(X)$ as a convergent power series in the $c_\Lambda(X)$ as follows. Define

$$d=H_{B_0}+\sum_{X\subset\partial\Lambda,\, X\cap B_0\neq\varnothing} c_\Lambda(X)\,\sigma(X) \tag{2.11}$$

Then $\|d\|\to 0$ as $\beta\to 0$. Now

$$\exp\left[\sum_{X\subset\partial\Lambda'} f_\Lambda(X)\,\sigma(X)\right]=\sum_{\{\sigma_i\}_{i\in B_0}} t(s_{B_0},\{\sigma_i\}_{i\in B_0})e^d=8(1+g)$$

where

$$g=\sum_{n=1}^{\infty}\frac{1}{8n!}\sum_{\{\sigma_i\}_{i\in B_0}} t(s_{B_0},\{\sigma_i\}_{i\in B_0})d^n \tag{2.12}$$

and we have used the fact that

$$\sum_{\{\sigma_i\}_{i\in B_0}} t(s_{B_0},\{\sigma_i\}_{i\in B_0})=8$$

Since we are in a Banach algebra, it follows that $\|g\|\to 0$ as $\|d\|\to 0$. So $\|g\|\to 0$ as $\beta\to 0$. If $\|g\|<1$, we can expand $\ln(1+g)$ in a power series to obtain

$$\sum_{X\subset\partial\Lambda'} f_\Lambda(X)\,\sigma(X)=\ln 8+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}g^n$$

The above equation is for the infinite-volume case. We can use it to define $f(X)$ in the infinite-volume limit. Let

$$d_\infty=H_{B_0}+\sum_{X\subset\partial\Lambda_\infty,\, X\cap B_0\neq\varnothing} c(X)\,\sigma(X)$$

and let g_∞ be given by (2.12) with d replaced by d_∞. Then we define $f(X)$ by

$$\sum_{X \subset \partial A'_\infty} f(X)\,\sigma(X) = \ln 8 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} g_\infty^n$$

Our bounds also imply that

$$\left\| \sum_{X \subset \partial A,\, X \cap B_0 \neq \varnothing} c_A(X)\,\sigma(X) - \sum_{X \subset \partial A_\infty,\, X \cap B_0 \neq \varnothing} c(X)\,\sigma(X) \right\| \to 0$$

as $L \to \infty$. So

$$\lim_{L \to \infty} \sum_{X \subset \partial A'} f_A(X)\,\sigma(X) = \sum_{X \subset \partial A'_\infty} f(X)\,\sigma(X)$$

We now let $L \to \infty$ in (2.6) and (2.7) and use (2.9) to obtain for $X \subset \partial A_\infty$

$$c(X) = c(X + (0,2)) + f_A(X + (0,2)), \qquad X + (0,2) \subset \partial A_\infty$$
$$c(X) = f(X + (0,2)), \qquad X + (0,2) \not\subset \partial A_\infty$$

The above equations involve $c(X)$ for sets X with $X \cap B_0 = \varnothing$ as well as sets with $X \cap B_0 \neq \varnothing$. Only sets X satisfying the second condition appear in the definition of $f(X)$. This suggests that we try to eliminate all the $c(X)$ with $X \cap B_0 = \varnothing$. This can be done,[9] and the end result is the following equation for X with $X \cap B_0 \neq \varnothing$:

$$c(X) = \sum_{n,m:X} f(X + (2n, 2m)) \tag{2.13}$$

The condition n, m: X means that n, m are summed over all integers such that $X + (2n, 2m) \subset \partial A'_\infty$. Recall that $f(X)$ depends on $c = \{c(X)\colon X \subset \partial A_\infty, X \cap B_0 \neq \varnothing\}$. We will denote the right side of (2.13) by $F(X, c)$ and denote the collection $\{F(X, c)\colon X \subset \partial A_\infty, X \cap B_0 \neq \varnothing\}$ by just $F(c)$, so that (2.13) may be written succinctly as $F(c) = c$. We should note that $F(c)$ also depends on β, since the definition of $f(X)$ involves H_{B_0}.

Thus far we have only been considering the block spin configuration in which $s_B = +1$ for all the blocks. We now consider the other two block spin configurations in Fig. 1. Equation (2.9) was true because when we shift the block spin configuration with all $s_B = +1$, it is unchanged. For the configurations in Figs. 1b and 1c, shifting the configuration by one block changes it, but the change is just given by a global spin flip. Thus the analog of (2.9) in these cases is

$$\lim_{L \to \infty} c_{A'(L)}(X) = -c(X - (0, 2)) \tag{2.14}$$

When one follows through the elimination of the $c(X)$ for X which do not intersect B_0, one finds that the analog of (2.13) for the block spin configurations of Figs. 1b and 1c is

$$c(X) = \sum_{n,m:X} \operatorname{sgn}(n, m)\, f(X + (2n, 2m)) \tag{2.15}$$

where $\operatorname{sgn}(n, m)$ is $+1$ or -1 as follows. Recall that B_0 is the block containing $(0, 0)$. Let $B_{(n,m)}$ be the block containing $(2n, 2m)$. The $\operatorname{sgn}(n, m) = +1$ if $s_{B_0} = s_{B_{(n,m)}}$, and $\operatorname{sgn}(n, m) = -1$ if $s_{B_0} \neq s_{B_{(n,m)}}$.

In order to use $F(c) = c$ as the definition of the infinite-volume limit even when β is not small, we need to define $F(c)$ in the case that $\|c\|$ and $\|H\|$ are not small. Note that in this case the power series approach we used to define $f(X)$ will not work. The needed definition has been carried out in ref. 9. The idea is the following. Suppose $c = \sum_X c(X)\, \sigma(X)$, where the sum only involves a finite number of X. Then (2.5) may be used as the definition of $f(X)$, since only a finite number of sites will appear in this equation. We then define $F(c + c')$ for c' with small $\|c'\|$ by doing an expansion around $F(c)$.

For small β the convergent high-temperature expansions show there is a solution of the fixed-point equation (2.13). We will show there is a solution for other values of β by finding an approximate solution and showing that $F(c)$ is a contraction in a large enough neighborhood of this approximate solution. Let $DF(c)$ denote the Jacobian of the map $F(c)$. Let $\|DF(c)\|$ denote the norm of this linear operator with respect to the norm (2.10) for the Banach space. For $A \subset B_0$, define

$$\langle \sigma(A) \rangle_0 = Z^{-1} \sum_{\{\sigma_i\}_{i \in B_0}} \sigma(A)\, t(s_{B_0}, \{\sigma_i\}_{i \in B_0}) \times \exp\left[H_{B_0} + \sum_{X \subset \partial A,\, X \cap B_0 \neq \varnothing} c_A(X)\, \sigma(X) \right]$$

where Z is chosen so that $\langle 1 \rangle_0 = 1$. [This "expectation" $\langle \cdot \rangle_0$ has nothing to do with the expectation $\langle \cdot \rangle_s$ defined by (1.3).]

Then $\langle \sigma(A) \rangle_0$ is a function of σ_i with $i \in \partial A'_\infty$. For the set of c for which $F(c)$ is defined, we have the expansion[9]

$$\langle \sigma(A) \rangle_0 = \sum_X d(A, X)\, \sigma(X) \tag{2.16}$$

where the sum is over subsets X of $\partial \Lambda'_\infty$. The techniques of ref. 9 show that

$$\|DF(c)\| \leqslant \max_{A : A \subset B_0, A \neq \varnothing} e^{-\mu(A)} \sum_X |d(A, X)| \, e^{\mu(X)}$$

If c only involves a finite number of sites, then there are only a finite number of nonzero $d(A, X)$, and so the above bound may be explicitly computed. For small perturbations of such finitely supported c we may use the following bound[9]:

$$\|DF(c + c')\| \leqslant \|DF(c)\| + \varepsilon(\|c'\|)$$

where $\varepsilon(x) = 2(e^x - 1)/(2 - e^x)$. By a standard fixed-point theorem, if we can find an approximate solution c_0 such that

$$\min_r \frac{\|F(c_0) - c_0\|}{r[1 - \|DF(c_0)\| - \varepsilon(r)]} < 1 \tag{2.17}$$

then the fixed-point equation $F(c) = c$ has an exact solution. (The min is only over r such that the denominator is positive.)

The above condition for the existence of an exact solution may be improved slightly. Define c_n inductively by $c_n = F(c_{n-1})$. We have

$$\|F(c_n) - c_n\| \leqslant \|F(c_{n-1}) - c_{n-1}\| \sup \|DF(c)\|$$

where the supremum is over c along the line segment between c_{n-1} and $F(c_{n-1})$. Using the above bound on $\|DF(c)\|$, this becomes

$$\|F(c_n) - c_n\| \leqslant \|F(c_{n-1}) - c_{n-1}\| \, [\|DF(c_{n-1})\| + \varepsilon(\|F(c_{n-1}) - c_{n-1}\|)] \tag{2.18}$$

We also have the bound

$$\|DF(c_n)\| \leqslant \|DF(c_{n-1})\| + \varepsilon(\|F(c_{n-1}) - c_{n-1}\|)] \tag{2.19}$$

Given bounds on $\|DF(c_0)\|$ and $\|F(c_0) - c_0\|$, we can iterate the above inequalities. If the bounds on $\|DF(c_n)\|$ converge to a number less than one, then the bounds on $\|F(c_n) - c_n\|$ converges to zero geometrically and we can prove that there is an exact fixed point.

Next we show how one extracts quantities like the free energy and correlation functions from the fixed-point equation. In the infinite-volume limit the free energy comes from the term in (2.1) that has no σ dependence, i.e., the $c_A(\varnothing)$ term. Thus from (2.4) we see that the infinite-volume limit of the free energy per site is given by $\frac{1}{4} f(\varnothing)$. (The factor of

1/4 enters because there are four sites in the block B_0.) Thus, to compute the free energy, one must first find the solution c of the fixed-point equation and then compute $f(\varnothing)$ using this c.

Next we consider the analyticity of the free energy in β. The function $f(\varnothing)$ depends explicitly on β since H_{B_0} depends on β. It also depends implicitly on β since the solution c of the fixed-point equation depends on β. Thus the derivative of the free energy with respect to β is given by 1/4 times

$$\frac{df(\varnothing)}{d\beta}=\frac{\partial f(\varnothing)}{\partial\beta}+\sum_{X\subset\partial A,\,X\cap B_0\neq\varnothing}\frac{\partial F(\varnothing)}{\partial c(X)}c'(X)$$

where c' denotes the derivative of the fixed-point equation solution with respect to β. Using the fixed-point equation $F(c)=c$, we find that it is given by

$$c'=(1-DF)^{-1}\frac{\partial F}{\partial\beta}$$

Substituting this in the previous equation and streamlining the notation, we obtain

$$\frac{df(\varnothing)}{d\beta}=\frac{\partial f(\varnothing)}{\partial\beta}+(DFc')(\varnothing)=\frac{\partial f(\varnothing)}{\partial\beta}+\left[DF(1-DF)^{-1}\frac{\partial F}{\partial\beta}\right](\varnothing) \tag{2.20}$$

To obtain the correlation function $\langle\sigma(A)\rangle_s$ we add the term $\alpha\sum_i\sigma(A+i)$ to the Hamiltonian and compute the derivative of the free energy per site with respect to α at $\alpha=0$. (The sum over i is over all lattice sites i, and $A+i$ denotes the translate of the set A by i.) The computation is completely analogous to the above computation. Since $\|DF\|<1$, we may rewrite the result as

$$\sum_{n=0}^{\infty}\left[(DF)^n\frac{\partial F}{\partial\alpha}\right](\varnothing) \tag{2.21}$$

To show that truncated correlation functions decay exponentially we need to develop a geometric description of the above equation. Recall that $F(X,c)$ was defined to be the right side of (2.13). Using the notation of (2.16), this yields

$$\frac{\partial F}{\partial\alpha}=\sum_X d(A,X)\,\sigma(X+t(X))$$

where $t(X)$ is the unique multiple of $(2, 2)$ such that $X + t(X)$ intersects B_0 and is a subset of ∂A_∞. This shift of $\sigma(X)$ to $\sigma(X + t(X))$ will appear frequently in the following, so we will simply refer to it as *the shift*. The above equation says that to compute $\partial F/\partial \alpha$ we first compute $\langle \sigma(A) \rangle_0$, write it in the form (2.16), and then apply the shift.

The action of DF may also be described using the shift. To compute the result of applying the linear transformation DF to $\sigma(X)$ we compute $\langle \sigma(X) \rangle_0$ and then apply the shift. Thus, Eq. (2.21) says that the correlation function $\langle \sigma(A) \rangle_s$ is computed as follows. Compute $\langle \sigma(A) \rangle_0$ and write the result as in (2.16). The σ-independent term $d(A, \varnothing)$ contributes to the correlation function. For all the other terms $d(A, X)\, \sigma(X)$ we apply the shift and then compute the $\langle \cdot \rangle_0$ of the shifted term. We write the result as $\sum_Y a(Y)\, \sigma(Y)$. The $Y = \varnothing$ term contributes to the correlation function. For all the other terms we repeat the procedure, i.e., we apply the shift and then take the $\langle \cdot \rangle_0$ expectation. The σ-independent term in the result goes into the correlation function. We should point out that since the expectation $\langle \cdot \rangle_0$ only involves the spins in B_0, for any X we have $\langle \sigma(X) \rangle_0 = \langle \sigma(X \cap B_0) \rangle_0\, \sigma(X \backslash B_0)$.

To show that truncated correlation functions decay exponentially, we replace the observable $\sigma(A)$ by $\sigma(A)\, \sigma(B)$ with A and B widely separated. We first give the argument for the special case that for each $X \subset B_0$ there are only finitely many nonzero terms in the expansion of $\langle \sigma(X) \rangle_0$. In this case when we compute $\langle \sigma(A)\, \sigma(B) \rangle_0$ with A and B widely separated, the terms in the result must be of the form $\sigma(A')\, \sigma(B')$, where the distance between A' and B' is at least as large as the distance between A and B minus some fixed distance. The shift does not change the relative distance between A' and B'. It is possible, however, that one but not both of A' or B' will be the empty set. When this happens the ensuing terms contribute to the correlation funuction $\langle \sigma(A) \rangle_s \langle \sigma(B) \rangle_s$. As we repeatedly apply DF, each application can move the two widely separated sets closer or it can wipe out one of the sets completely. The terms from the latter case contribute to $\langle \sigma(A) \rangle_s \langle \sigma(B) \rangle_s$, and so are not part of the truncated correlation function $\langle \sigma(A); \sigma(B) \rangle_s$. As the sets move closer together they can eventually intersect. The number of applications of DF needed to achieve this is proportional to the distance between A and B. Since $\|DF\| < 1$, this implies that the sum of such terms is exponentially small in the distance between A and B.

Unfortunately, in general there will be infinitely many nonzero terms in the expansion of $\langle \sigma(X) \rangle_0$ for $X \subset B_0$. Thus it is possible that a single application of DF can make the two widely separated sets intersect. The terms in (2.16) that can cause this to occur come from sets X which contain at least one site that is very far from the block B_0. The coefficient of such a term should be very small. To quantify this argument we must modify the definition of the norm slightly. The new norm is

$$\|c\| = \sum_{X} |c(X)|\, e^{\mu(X) + \varepsilon d(X)}$$

where $d(X)$ is the maximum distance between two sites in X or between a site in the block B_0 and a site in X. ε is a small positive number. If we have a finitely supported approximate fixed point for which the condition for the existence of an exact fixed point is satisfied when we use the original norm, then the condition is also satisfied when we use the new norm if ε is sufficiently small. In this new norm a term in the expansion of $\langle \sigma(X) \rangle_0$ which can move the sets A and B closer by a distance l is weighted by a factor at least as large as $e^{\varepsilon l}$. This leads to the exponential decay of the truncated correlation functions.

3. THE COMPUTER CALCULATIONS

In the previous section we derived a sufficient condition for the existence of a solution of the fixed-point equation (2.13) which we use to define the infinite-volume limit. The next step is to find an approximate solution of the fixed-point equation that satisfies this condition. Since the approximate solution will only depend on a finite number of spins, checking whether or not it satisfies the condition is a computation that only involves a finite number of spins, and so may be done on a computer. Even if the approximate solution only involves a modest number of spins, e.g., 15, the computations required are nontrivial and must be done in a reasonably efficient manner. We can greatly reduce the amount of computation required with the following observation. Since the condition for the existence of a solution of the fixed-point equation is that some quantity be strictly less than 1, we never need to compute anything exactly. Sufficiently accurate bounds on quantities will do.

Consider storing a function $f(\sigma)$ of the spins σ_i for $i \in V$. We could store the value of the function for each spin configuration σ. If $|V|$ is the number of sites in V, there will be $2^{|V|}$ values to be stored. We could also write the function in the form

$$f(\sigma) = \sum_{X} c(X)\, \sigma(X) \tag{3.1}$$

where X is summed over all subsets of V and store the $2^{|V|}$ coefficients $c(X)$. For the functions that we typically encounter, many of these coefficients are very small, so we can restrict the sum over X to a small subcollection of the collection of subsets of V and still approximate $f(\sigma)$ quite well. Thus what we actually store is the following: a collection $X_1,\ldots, X_n$ of subsets of V, coefficients c_i for $i=1,\ldots, n$, and an "error" ε which is a positive number such that

$$\left\| f(\sigma) - \sum_{i=1}^{n} c_i \sigma(X_i) \right\| \leqslant \varepsilon \tag{3.2}$$

The main operations we must carry out on these functions are addition, multiplication, exponentiation, and logarithms.

Addition is easy. If

$$\begin{aligned} \left\| f(\sigma) - \sum_{i=1}^{n} c_i \sigma(X_i) \right\| &\leqslant \varepsilon \\ \left\| g(\sigma) - \sum_{j=1}^{m} d_j \sigma(Y_j) \right\| &\leqslant \delta \end{aligned} \tag{3.3}$$

then

$$\left\| f(\sigma) + g(\sigma) - \sum_{i=1}^{n} c_i \sigma(X_i) - \sum_{j=1}^{m} d_j \sigma(Y_j) \right\| \leqslant \varepsilon + \delta$$

Of course some of the X_i may equal some of the Y_j, in which case we consolidate the two terms into a single term in the list for $f(\sigma)+g(\sigma)$.

Next we consider multiplication of two functions $f(\sigma)$ and $g(\sigma)$. Suppose that the inequalities in (3.3) holds. Then we have

$$\left\| f(\sigma)\, g(\sigma) - \sum_{k=1}^{r} p_k \sigma(Z_k) \right\| \leqslant C\varepsilon + D\delta + \varepsilon\delta \tag{3.4}$$

where

$$C = \sum_{i=1}^{n} |c_i|\, e^{\mu(X_i)}$$

and D is defined similarly. [Recall that $\mu(X)=\sum_{i\in X}\mu_i$.] The list Z_k includes all the sets of the form $X_i \triangle Y_j$, and the coefficients p_k are given by

$$p_k = \sum_{i,j:X_i \triangle Y_j = Z_k} c_i d_j \tag{3.5}$$

[Recall that $X \triangle Y$ is the set of sites in X or Y but not both. So $\sigma(X)\,\sigma(Y)=\sigma(X \triangle Y)$.] Typically, n and m will be large, but many of the coefficients will be quite small. This suggests that we only compute a subset of the terms in the above formula for p_k. There are various criteria one could adopt for deciding which terms to keep. We use the following. Let $\gamma > 0$. Define

$$I = \{(i, j): 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ |c_i d_j| \geqslant \gamma\}$$
$$I^c = \{(i, j): 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ |c_i d_j| < \gamma\}$$

Then we let

$$\tilde{p}_k = \sum_{(i, j) \in I: X_i \,\triangle\, Y_j = Z_k} c_i d_j \tag{3.6}$$

We then must add the following quantity to the error term for $f(\sigma)\, g(\sigma)$:

$$\sum_{(i, j) \in I^c} |c_i d_j|\, e^{\mu(X_i) + \mu(Y_j)} \tag{3.7}$$

The most time-consuming operations are exponentiation and taking the logarithm. To compute the former we make use of the following identity:

$$\exp\left[\sum_{i=1}^{n} c_i \sigma(X_i)\right] = \prod_{i=1}^{n} \cosh(c_i) \prod_{i=1}^{n} [1 + \sigma(X_i) \tanh(c_i)] \tag{3.8}$$

This reduces the computation to a sequence of multiplications. We do these using the method discussed in the previous paragraph. Each multiplication has some error, and these errors propagate through the calculation, but we always have rigorous bounds on them.

To compute logarithms, the following algorithm works well, although at first glance it looks like a disaster. By multiplying our function by an overall constant, it is enough to consider computing

$$\log\left[1 + \sum_{X \neq \varnothing} c(X)\, \sigma(X)\right]$$

where the sum over X is over some finite collection of sets of sites. Pick Y such that $|c(Y)| = \sup_{X \neq \varnothing} |c(X)|$. We then use

$$\log\left[1 + \sum_{X \neq \varnothing} c(X)\, \sigma(X)\right] = \log\left[\sum_X \tilde{c}(X)\, \sigma(X)\right] - \log[1 - c(A)\, \sigma(A)] \tag{3.9}$$

where

$$\sum_X \tilde{c}(X)\, \sigma(X) = \left[1 + \sum_{X \neq \varnothing} c(X)\, \sigma(X)\right] [1 - c(A)\, \sigma(A)] \tag{3.10}$$

The second log in the right side of (3.9) is trivial to compute, since $\sigma(A)$ can only be $+1$ or -1. In computing the product in the right side of (3.10), there will be some cancellation; the term $c(A)\,\sigma(A)$ and its opposite appear. This product will also generate a lot of other terms. However, we have found in practice that the norm of $\sum_X \tilde{c}(X)\,\sigma(X)$ is smaller than that of $1+\sum_{X\neq\varnothing} c(X)\,\sigma(X)$. We iterate the above process until the norm of $\sum_{X\neq\varnothing} c(X)\,\sigma(X)$ is small. If it is small enough, then $\log[1+\sum_{X\neq\varnothing} c(X)\,\sigma(X)]$ has a convergent power series. So we can approximate it by $\sum_{X\neq\varnothing} c(X)\,\sigma(X)$ and find a rigorous bound on the error. The function we start with will usually contain some error term, and the use of our multiplication algorithm to compute the right side of Eq. (3.10) will produce some error. We must compute bounds on these errors as we iterate the above procedure. At the end of the iteration, in order for $\log[1+\sum_{X\neq\varnothing} c(X)\,\sigma(X)]$ to have a convergent power series, the sum of the norm of $\sum_{X\neq\varnothing} c(X)\,\sigma(X)$ and the error for this function must be small.

Now we turn to the numerical results. We must do two things. First we must compute an approximate solution c_0 to the fixed-point equation. Then we must compute bounds on $\|F(c_0)-c_0\|$ and $\|DF(c_0)\|$. To find an approximate solution to $F(c)=c$, we truncate it so it becomes a finite-dimensional equation. We do this as follows. Let V be a finite subset of the right boundary of ∂A, and let $c=\sum_{X\subset V} c(X)\,\sigma(X)$ be a function of the spins in V. Because of the shifting involved, $F(c)$ will not be function of just the spins in V. We will have

$$F(c)=\sum_X F(X, c)\,\sigma(X)$$

where some of the sets X are not subsets of V. We truncate this by defining

$$F_V(c)=\sum_{X\subset V} F(X, c)\,\sigma(X)$$

In other words, we simply drop the terms which are not supported on V after the shifting.

The fixed-point equation $F_V(c)=c$ is finite dimensional. We solve it by iteration and take c_0 to be its solution. [Actually, we make one further truncation. In $F_V(c)$ we keep only the terms with four or less sites.] We then compute bounds on $\|F(c_0)-c_0\|$ and $\|DF(c_0)\|$ using the algorithms discussed above. The μ_i are chosen to chosen to minimize the left side of (2.17). For each of the three block spins configurations of Fig. 1, we use different choices for the set V. In all three cases V contains the sites (0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (0, 3), (0, 4), (0, 5), (2, -1), (2, -2), (2, -3),

Table I. Results for the Block Spin Configuration of Fig. 1[a]

	$\|F(c_0)-c_0\|_\mu$	$\|DF(c_0)\|_\mu$	$\|DF(c)\|_\mu$	$\mu_{(0,0)}$	$\mu_{(1,0)}$	$\mu_{(0,1)}$	$\mu_{(1,1)}$
a	0.005221	0.840130	0.9358	0.2	0.56	0.45	0.0
b	0.003147	0.885046	0.9813	0.0	0.24	0.2	0.0
c	0.003242	0.883687	0.9836	0.2	0.39	0.38	0.0

[a] For each of the three block spin configurations shown in Figs. 1a–1c we find an approximate fixed point c_0 and compute upper bounds on $\|F(c_0)-c_0\|_\mu$ and $\|DF(c_0)\|_\mu$. These bounds are used in inequalities (2.18) and (2.19) to obtain a bound on $\|DF(c)\|_\mu$, where c is the exact solution of the fixed-point equation.

(2, −4), and (2, −5). In addition, for the block spin configuration in Fig. 1b, V contains the sites (0, 6) and (2, −6), and for the configuration in Fig. 1c it contains (2, −6). The bounds we find are shown in Table I. We iterate (2.18) and (2.19) and show the resulting bound on $\|DF(c)\|$ at the exact fixed point in the third column. Remember that this quantity must be less than one to conclude there is an exact fixed point. We should caution the reader that since we have used different V's and different values for the μ_i for the three block spin configuration, comparisons between the rows of the table are rather meaningless. These calculations are carried out at β_c, the critical temperature of the unconditioned model. Our approximate fixed point only involves a finite number of sites, so $\|F(c_0)-c_0\|$ and $\|DF(c_0)\|$ are continuous in β. Thus if the condition holds at β_c, then it holds in a neighborhood of β_c.

ACKNOWLEDGMENTS

It is a pleasure to thank the authors of ref. 13 for supplying a copy of their work prior to publication and for useful discussions. This work was supported in part by NSF grant DMS-9103621.

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